Heat Equation Practice Problems

Solutions.

1) Solve $u_t = u_{xx}$ $\quad u(0,t) = 0 = u(1,t)$

& $f(x) = u(x,0) = (1-x) \quad 0 < x < 1.$

We see in this question that $L = 1$
& $\alpha^2 = 1$

& thus, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-(n\pi)^2 t}$$

where $b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx \qquad n = 1, 2, 3, \ldots$

$$= 2\int_0^1 \underbrace{(1-x)}_{u}\,\underbrace{\sin(n\pi x)\,dx}_{dv}$$

$u = 1-x \qquad dv = \sin(n\pi x)\,dx$

$\Rightarrow du = -dx \qquad \rightarrow v = \frac{-\cos(n\pi x)}{n\pi}$

$$\Rightarrow b_n = 2\left(-(1-x)\frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 - \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx$$

$$\rightarrow b_n = 2\left(+1\,\frac{1}{n\pi}\right) - \frac{\sin n\pi x}{(n\pi)^2}\Bigg|_0^1 = +\,{}^{2}/_{n\pi}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x)\, e^{-(n\pi)^2 t}$$

2) $u_t = 4u_{xx}$ $\quad u(0,t) = 0 = u(2,t)$

$f(x) = u(x,0) = -(x-1)^2 + 1$ for $0 < x < 2$

i.e $L = 2$ & $\alpha^2 = 4$. Thus soln is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-4\left(\frac{n\pi}{2}\right)^2 t}$$

where $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

$$\Rightarrow b_n = \int_0^2 \underbrace{\left(-(x-1)^2+1\right)}_{u} \underbrace{\sin\left(\frac{n\pi x}{2}\right)dx}_{dv}$$

$u = -(x-1)^2 + 1 \qquad dv = \sin\left(\frac{n\pi x}{2}\right)dx \Rightarrow v = \frac{-2\cos\left(\frac{n\pi x}{2}\right)}{(n\pi)}$

$\Rightarrow du = (-2(x-1))\,dx$

$$\Rightarrow b_n = \left(-(x-1)^2+1\right)\frac{-2\cos\left(\frac{n\pi x}{2}\right)}{(n\pi)}\Bigg|_0^2 - \int_0^2 \frac{-2\cos\left(\frac{n\pi x}{2}\right)}{(n\pi)} -2(x-1)\,dx$$

$$\Rightarrow b_n = -\int_0^2 \frac{4}{(n\pi)} \underbrace{(x-1)}_{u} \underbrace{\cos\left(\frac{n\pi x}{2}\right)dx}_{dv}$$

$u = x-1 \Rightarrow du = dx$, $dv = \cos\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow b_n = -\frac{4}{n\pi}\left((x-1)\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx \right)$$

$$= \frac{8}{(n\pi)^2}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx = \frac{-8}{(n\pi)^2}\frac{2}{(n\pi)}\cos\frac{n\pi x}{2}\Big|_0^2$$

$$\Rightarrow b_n = \frac{-16}{(n\pi)^3}\left(\cos n\pi - 1\right) = \frac{-16}{(n\pi)^3}\left((-1)^n - 1\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{16-16(-1)^n}{(n\pi)^3}\sin\left(\frac{n\pi x}{2}\right)e^{-4\left(\frac{n\pi}{2}\right)^2 t}.$$

3) $u_t + \frac{1}{2}u_{xx}$ (i.e. $\alpha^2 = 1/2$)

$u(0,t) = 0 = u(2,t)$ $(L=2)$

$$f(x) = u(x,0) = \begin{cases} x & 0 \le x \le 1 \\ 2x - x^2 & 1 \le x \le 2 \end{cases}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{1}{2}\left(\frac{n\pi}{2}\right)^2 t}$$

where $\displaystyle b_n = \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$

$$\Rightarrow b_n = \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2x - x^2) \sin\left(\frac{n\pi x}{2}\right) dx.$$

Now, let's calculate these integrals seperately using integration by parts

$$\int_0^1 \underbrace{x}_{u} \underbrace{\sin\left(\frac{n\pi x}{2}\right) dx}_{dv}$$

$u = x \Rightarrow du = dx$

$dv = \sin\left(\frac{n\pi x}{2}\right) dx$

$\Rightarrow v = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx = \left(\frac{-2x}{n\pi} \cos \frac{n\pi x}{2}\right)\Bigg|_0^1 + \int_0^1 \frac{2}{n\pi} \cos \frac{n\pi x}{2}\, dx$$

$$= \frac{-2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin \frac{n\pi x}{2}\Bigg|_0^1\right]$$

$$= \frac{-2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{(n\pi)^2} \sin \frac{n\pi}{2}.$$

$$\& \int_1^2 \underbrace{(2x-x^2)}_{u} \underbrace{\sin\left(\frac{n\pi x}{2}\right)dx}_{dv}$$

$u = 2x - x^2$

$\Rightarrow du = (2-2x)dx$

$dv = \sin\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow v = \frac{-2}{(n\pi)}\cos\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow \left((2x-x^2)\left(\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right)\right)\Bigg|_1^2 + \int_1^2 \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)(2-2x)\,dx$$

$$= 1\left(\frac{+2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) + \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)\underbrace{(2-2x)}_{u}\,dx$$

dv

$u = 2-2x \Rightarrow du = -2dx$

$dv = \cos\left(\frac{n\pi x}{2}\right) \Rightarrow v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\left((2-2x)\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Bigg|_1^2 + \int_1^2 2\sin\left(\frac{n\pi x}{2}\right)dx\right)$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\left(2\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx\right)$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n\pi}\cdot\frac{2}{(n\pi)}\left(-\cos\frac{n\pi x}{2}\right)\Bigg|_1^2$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{8}{(n\pi)^2}\cos n\pi + \frac{8}{(n\pi)^2}\cos\frac{n\pi}{2}.$$

Thus, combining these two integrals, we get

$$b_n = \cancel{\frac{-2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2} + \cancel{\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)} - \frac{8}{(n\pi)^2}\cos n\pi$$

$$+ \frac{8}{(n\pi)^2}\cos\frac{n\pi}{2}.$$

Then, we if $n = 4k$, $b_n = 0$

$$n = 4k+1 \Rightarrow b_{4k+1} = \frac{4}{(4k+1)^2\pi^2} + \frac{8}{(4k+1)^2\pi^2} = \frac{12}{(4k+1)^2\pi^2}$$

$$n = 4k+2 \Rightarrow b_{4k+2} = \frac{-8}{(4k+2)^2\pi^2} - \frac{8}{(4k+2)^2\pi^2} = \frac{-16}{(4k+2)^2\pi^2}$$

$$n = 4k+3 \Rightarrow b_{4k+3} = \frac{-4}{(4k+3)^2\pi^2} + \frac{8}{(4k+3)^2\pi^2} = \frac{4}{(4k+3)^2\pi^2}$$

4) $u_t = u_{xx}$ (i.e. $\alpha^2 = 1$)

$u(0,t) = 0 = u(2,t)$ $(L=2)$

$f(x) = u(x,0) = \sin\left(\frac{\pi x}{2}\right)$ $0 < x < 2$

Again, we can first find b_n's.

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\Rightarrow b_n = \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) dx$$

first let's calculate b_1.

$$b_1 = \int_0^2 \left(\sin \frac{\pi x}{2}\right)^2 dx = \frac{1}{2}\int_0^2 1 - \cos\left(\frac{2\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^2 (1 - \cos \pi x) dx = \frac{1}{2}\left(x - \frac{\sin \pi x}{\pi}\right)\Big|_0^2$$

$$= 1$$

And if $n \neq 1$, using the identity

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right),$$

we get

$$b_n = \frac{1}{2}\int_0^2 \cos\left((n-1)\frac{\pi x}{2}\right) - \cos\left((n+1)\frac{\pi x}{2}\right) dx$$

$$= \frac{1}{2}\left[\frac{\sin(n-1)\frac{\pi x}{2}}{(n-1)\pi/2} - \frac{\sin\left((n+1)\frac{\pi x}{2}\right)}{(n+1)\pi/2}\right]_0^2 = 0.$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{2}\right)^2 t}$$

$$= b_1 \sin\left(\frac{\pi x}{2}\right) e^{-\left(\frac{\pi}{2}\right)^2 t} + 0 + 0 \cdots$$

$$= \sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2}{4} t}.$$